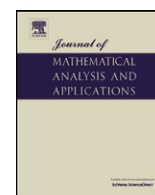


Contents lists available at [SciVerse ScienceDirect](http://SciVerse.ScienceDirect.com)

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Periodic solutions for nonlinear evolution equations at resonance[☆]

Piotr Kokocki

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland

ARTICLE INFO

Article history:

Received 2 September 2010

Available online 5 March 2012

Submitted by P.J. McKenna

Keywords:

Semigroup

Evolution equation

Topological degree

Periodic solution

Resonance

ABSTRACT

We are concerned with periodic problems for nonlinear evolution equations at resonance of the form $\dot{u}(t) = -Au(t) + F(t, u(t))$, where a densely defined linear operator $A : D(A) \rightarrow X$ on a Banach space X is such that $-A$ generates a compact C_0 semigroup and $F : [0, +\infty) \times X \rightarrow X$ is a nonlinear perturbation. Imposing appropriate Landesman–Lazer type conditions on the nonlinear term F , we prove a formula expressing the fixed point index of the associated translation along trajectories operator, in the terms of a time averaging of F restricted to $\text{Ker } A$. By the formula, we show that the translation operator has a nonzero fixed point index and, in consequence, we conclude that the equation admits a periodic solution.

© 2012 Elsevier Inc. All rights reserved.

1. Introduction

Consider a periodic problem

$$\begin{cases} \dot{u}(t) = -Au(t) + F(t, u(t)), & t > 0, \\ u(t) = u(t + T), & t \geq 0, \end{cases} \quad (1.1)$$

where $T > 0$ is a fixed period, $A : D(A) \rightarrow X$ is a linear operator such that $-A$ generates a C_0 semigroup of bounded linear operators on a Banach space X and $F : [0, +\infty) \times X \rightarrow X$ is a continuous mapping. The periodic problems are the abstract formulations of many differential equations including the parabolic partial differential equations on an open set $\Omega \subset \mathbb{R}^n$, with smooth boundary, of the form

$$\begin{cases} u_t = -\mathcal{A}u + f(t, x, u) & \text{in } (0, +\infty) \times \Omega, \\ \mathcal{B}u = 0 & \text{on } [0, +\infty) \times \partial\Omega, \\ u(t, x) = u(t + T, x) & \text{in } [0, +\infty) \times \Omega, \end{cases} \quad (1.2)$$

where

$$\mathcal{A}u = -D_i(a^{ij}D_j u) + a^k D_k u + a_0 u$$

is such that $a^{ij} = a^{ji} \in C^1(\overline{\Omega})$, $a^k, a_0 \in C(\overline{\Omega})$,

$$a^{ij}(x)\xi_i \xi_j \geq \theta |\xi|^2 \quad \text{for } \xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n, x \in \Omega,$$

\mathcal{B} stands for the Dirichlet or Neumann boundary operator and $f : [0, +\infty) \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous mapping.

Given $x \in X$, let $u(t; x)$ be a (mild) solution of

$$\dot{u}(t) = -Au(t) + F(t, u(t)), \quad t > 0$$

[☆] The researches supported by the MNISzW Grant No. N N201 395137.

E-mail address: p.kokocki@mat.umk.pl.

such that $u(0; x) = x$. We look for the T -periodic solutions of (1.1) as the fixed points of the translation along trajectory operator $\Phi_T: X \rightarrow X$ given by $\Phi_T(x) := u(T; x)$.

One of the effective methods used to prove the existence of the fixed points of Φ_T is the averaging principle involving the equations

$$\dot{u}(t) = -\lambda Au(t) + \lambda F(t, u(t)), \quad t > 0 \quad (1.3)$$

where $\lambda > 0$ is a parameter. Let $\Theta_T^\lambda: X \rightarrow X$ be the translation operator for (1.3). It is clear that $\Phi_T = \Theta_T^1$. Define the mapping $\widehat{F}: X \rightarrow X$ by $\widehat{F}(x) := \frac{1}{T} \int_0^T F(s, x) ds$ for $x \in X$. The averaging principle says that for every open bounded set $U \subset X$ such that $0 \notin (-A + \widehat{F})(D(A) \cap \partial U)$, one has that $\Theta_T^\lambda(x) \neq x$ for $x \in \partial U$ and

$$\deg(I - \Theta_T^\lambda, U) = \deg(-A + \widehat{F}, U)$$

provided $\lambda > 0$ is sufficiently small. In the above formula \deg stands for the appropriate topological degree. Therefore, if $\deg(-A + \widehat{F}, U) \neq 0$, then using suitable *a priori* estimates and the continuation argument, we infer that Θ_T^1 has a fixed point and, in consequence, (1.1) admits a periodic solution starting from \bar{U} . The averaging principle for periodic problems on finite dimensional manifolds was studied in [13]. The principle for the equations on any Banach space has been recently considered in [5] in the case when $-A$ generates a compact C_0 semigroup and in [6] for A being an m -accretive operator. In [8], a similar results were obtained when $-A$ generates a semigroup of contractions and F is condensing. For the results when the operator A is replaced by a time-dependent family $\{A(t)\}_{t \geq 0}$ see [9].

However there are examples of equations where the averaging principle in the above form is not applicable. Therefore, in this paper, motivated by [3,1,14,18], we use the method of translation along trajectories operator to derive its counterpart in the particular situation when Eq. (1.1) is at resonance i.e., $\text{Ker } A \neq 0$ and F is bounded. Let $N := \text{Ker } A$ and assume that the C_0 semigroup $\{S_A(t)\}_{t \geq 0}$ generated by $-A$ is compact. Then it is well known that (real) eigenvalues of $S_A(T)$ make a sequence which is either finite or converges to 0 and the algebraic multiplicity of each of them is finite. Denote by μ the sum of the algebraic multiplicities of eigenvalues of $S_A(T): X \rightarrow X$ lying in $(1, +\infty)$. Since the semigroup is compact, the operator A has compact resolvents and, in consequence, $\dim N < +\infty$. Let M be a subspace of X such that $N \oplus M = X$ with $S_A(t)M \subset M$ for $t \geq 0$. Define a mapping $g: N \rightarrow N$ by

$$g(x) := \int_0^T P F(s, x) ds \quad \text{for } x \in N \quad (1.4)$$

where $P: X \rightarrow X$ is a topological projection onto N with $\text{Ker } P = M$.

First, we are concerned with an equation

$$\dot{u}(t) = -Au(t) + \varepsilon F(t, u(t)), \quad t > 0$$

where $\varepsilon \in [0, 1]$ is a parameter. Denoting by $\Phi_t^\varepsilon: X \rightarrow X$ the translation along trajectory operator associated with this equation, we shall show that, if $V \subset M$ is an open bounded set, with $0 \in V$ and $U \subset N$ is an open bounded set in N such that $g(x) \neq 0$ for x from the boundary $\partial_N U$ of U in N , then for small $\varepsilon \in (0, 1)$, $\Phi_T^\varepsilon(x) \neq x$ for $x \in \partial(U \oplus V)$ and

$$\deg_{\text{LS}}(I - \Phi_T^\varepsilon, U \oplus V) = (-1)^{\mu + \dim N} \deg_B(g, U). \quad (1.5)$$

Here \deg_{LS} and \deg_B stand for the Leray–Schauder and Brouwer degree, respectively. The obtained result improves that from [18].

Further, for an open and bounded set $\Omega \subset \mathbb{R}^n$, we shall use the formula (1.5) to study the periodic problem

$$\begin{cases} \dot{u}(t) = -Au(t) + \lambda u(t) + F(t, u(t)), & t > 0, \\ u(t) = u(t + T), & t \geq 0, \end{cases} \quad (1.6)$$

where $A: D(A) \rightarrow X$ is a linear operator on the Hilbert space $X := L^2(\Omega)$ with a real eigenvalue λ and $F: [0, +\infty) \times X \rightarrow X$ is a continuous mapping. As before we assume that $-A$ generates a compact C_0 semigroup $\{S_A(t)\}_{t \geq 0}$ on X . The mapping F is associated with a bounded and continuous $f: [0, +\infty) \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ as follows

$$F(t, u)(x) := f(t, x, u(x)) \quad \text{for } t \in [0, +\infty), x \in \Omega. \quad (1.7)$$

Additionally we suppose that the following kernel coincidence holds true (which is more general than to assume that A is self-adjoint)

$$N_\lambda := \text{Ker}(A - \lambda I) = \text{Ker}(A^* - \lambda I) = \text{Ker}(I - e^{\lambda T} S_A(T)).$$

Let $\Psi_t: X \rightarrow X$ be the translation along trajectories operator associated with the equation

$$\dot{u}(t) = -Au(t) + \lambda u(t) + F(t, u(t)), \quad t > 0.$$

The formula (1.5), under suitable Landesman–Lazer type conditions introduced in [16], gives an effective criterion for the existence of T -periodic solutions of (1.6). Namely, we prove that there is an $R > 0$ such that $g(x) \neq 0$ for $x \in N_\lambda \setminus B(0, R)$, $\Psi_T(x) \neq x$ for $x \in X \setminus B(0, R)$ and

$$\deg_{LS}(I - \Psi_T, B(0, R)) = (-1)^{\mu(\lambda) + \dim N_\lambda} \deg_B(g, B(0, R) \cap N_\lambda) \quad (1.8)$$

where $\mu(\lambda)$ is the sum of the algebraic multiplicities of the eigenvalues of $e^{\lambda T} S_A(T)$ lying in $(1, +\infty)$ and $g: N_\lambda \rightarrow N_\lambda$ is given by (1.4) with P being the orthogonal projection on N_λ . Additionally, we compute $\deg_B(g, B(0, R) \cap N_\lambda)$, which may be important in the study of problems concerning to the multiplicity of periodic solutions. Obtained applications correspond to those from [3,14], where a different approach were used to prove the existence of periodic solutions for parabolic equations at resonance. For the results concerning hyperbolic equations see e.g. [7,4,12].

Notation and terminology Throughout the paper we use the following notational conveniences. If $(X, \|\cdot\|)$ is a normed linear space, $Y \subset X$ is a subspace and $U \subset Y$ is a subset, then by $\text{cl}_Y U$ and $\partial_Y U$ we denote the closure and boundary of U in Y , respectively, while by $\text{cl} U$ (\bar{U}) and ∂U we denote the closure and boundary of U in X , respectively. If Z is a subspace of X such that $X = Y \oplus Z$, then for subsets $U \subset Y$ and $V \subset Z$ we write $U \oplus V := \{x + y \mid x \in U, y \in V\}$ for their algebraic sum. We recall also that a C_0 semigroup $\{S(t): X \rightarrow X\}_{t \geq 0}$ is compact if $S(t)V$ is relatively compact for every bounded $V \subset X$ and $t > 0$.

2. Translation along trajectories operator

Consider the following differential problem

$$\begin{cases} \dot{u}(t) = -Au(t) + F(\lambda, t, u(t)), & t > 0, \\ u(0) = x \end{cases} \quad (2.9)$$

where λ is a parameter from a metric space Λ , $A: D(A) \rightarrow X$ is a linear operator on a Banach space $(X, \|\cdot\|)$ and $F: \Lambda \times [0, +\infty) \times X \rightarrow X$ is a continuous mapping. In this section X is assumed to be real, unless otherwise stated. Suppose that $-A$ generates a compact C_0 semigroup $\{S_A(t)\}_{t \geq 0}$ and the mapping F is such that

(F1) for any $\lambda \in \Lambda$ and $x_0 \in X$ there is a neighborhood $V \subset X$ of x_0 and a constant $L > 0$ such that for any $x, y \in V$

$$\|F(\lambda, t, x) - F(\lambda, t, y)\| \leq L\|x - y\| \quad \text{for } t \in [0, +\infty);$$

(F2) there is a continuous function $c: [0, +\infty) \rightarrow [0, +\infty)$ such that

$$\|F(\lambda, t, x)\| \leq c(t)(1 + \|x\|) \quad \text{for } \lambda \in \Lambda, t \in [0, +\infty), x \in X.$$

A mild solution of the problem (2.9) is, by definition, a continuous mapping $u: [0, +\infty) \rightarrow X$ such that

$$u(t) = S_A(t)x + \int_0^t S_A(t-s)F(\lambda, s, u(s))ds \quad \text{for } t \geq 0.$$

It is well known (see e.g. [17]) that for any $\lambda \in \Lambda$ and $x \in X$, there is unique mild solution $u(\cdot; \lambda, x): [0, +\infty) \rightarrow X$ of (2.9) such that $u(0; \lambda, x) = x$ and therefore, for any $t \geq 0$, one can define the translation along trajectories operator $\Phi_t: \Lambda \times X \rightarrow X$ by

$$\Phi_t(\lambda, x) := u(t; \lambda, x) \quad \text{for } \lambda \in \Lambda, x \in X.$$

As we need the continuity and compactness of Φ_t , we recall the following

Theorem 2.1. Let $A: D(A) \rightarrow X$ be a linear operator such that $-A$ generates a compact C_0 semigroup and let $F: \Lambda \times [0, +\infty) \times X \rightarrow X$ be a continuous mapping such that conditions (F1) and (F2) hold.

(a) If sequences (λ_n) in Λ and (x_n) in X are such that $\lambda_n \rightarrow \lambda_0$ and $x_n \rightarrow x_0$, as $n \rightarrow +\infty$, then

$$u(t; \lambda_n, x_n) \rightarrow u(t; \lambda_0, x_0) \quad \text{as } n \rightarrow +\infty,$$

uniformly for t from bounded intervals in $[0, +\infty)$.

(b) For any $t > 0$, the operator $\Phi_t: \Lambda \times X \rightarrow X$ is completely continuous, i.e. $\Phi_t(\Lambda \times V)$ is relatively compact, for any bounded $V \subset X$.

Remark 2.2. The above theorem is slightly different from Theorem 2.14 in [5], where it is proved in the case when linear operator is dependent on parameter as the mapping F , and moreover the parameter space Λ is compact. The above theorem says that if A is free of parameters, then compactness of Λ may be omitted.

Before we start the proof we prove the following technical lemma:

Lemma 2.3. Let $\Omega \subset X$ be a bounded set. Then

- (a) for every $t_0 > 0$ the set $\{u(t; \lambda, x) \mid t \in [0, t_0], \lambda \in \Lambda, x \in \Omega\}$ is bounded;
 (b) for every $t_0 > 0$ and $\varepsilon > 0$ there is $\delta > 0$ such that if $t, t' \in [0, t_0]$ and $0 < t' - t < \delta$, then

$$\left\| \int_t^{t'} S_A(t' - s) F(\lambda, s, u(s; \lambda, x)) ds \right\| \leq \varepsilon \quad \text{for } \lambda \in \Lambda, x \in \Omega;$$

- (c) for every $t_0 > 0$ the set

$$S(t_0) := \left\{ \int_0^{t_0} S_A(t_0 - s) F(\lambda, s, u(s; \lambda, x)) ds \mid \lambda \in \Lambda, x \in \Omega \right\}$$

is bounded.

Proof. Throughout the proof we assume that the constants $M \geq 1$ and $\omega \in \mathbb{R}$ are such that $\|S_A(t)\| \leq Me^{\omega t}$ for $t \geq 0$.

- (a) Let $R > 0$ be such that $\Omega \subset B(0, R)$. Then by condition (F2), for every $t \in [0, t_0]$

$$\begin{aligned} \|u(t; \lambda, x)\| &\leq \|S_A(t)x\| + \int_0^t \|S_A(t-s)F(\lambda, s, u(s; \lambda, x))\| ds \\ &\leq Me^{|\omega|t} \|x\| + \int_0^t Me^{|\omega|(t-s)} c(s) (1 + \|u(s; \lambda, x)\|) ds \\ &\leq RMe^{|\omega|t_0} + t_0 KMe^{|\omega|t_0} + \int_0^t KMe^{|\omega|t_0} \|u(s; \lambda, x)\| ds, \end{aligned}$$

where $K := \sup_{s \in [0, t_0]} c(s)$. By the Gronwall inequality

$$\|u(t; \lambda, x)\| \leq C_0 e^{t_0 C_1} \quad \text{for } t \in [0, t_0], \lambda \in \Lambda, x \in \Omega, \quad (2.10)$$

where $C_0 := RMe^{|\omega|t_0} + t_0 KMe^{|\omega|t_0}$ and $C_1 := KMe^{|\omega|t_0}$.

- (b) From (a) it follows that there is $C > 0$ such that $\|u(t; \lambda, x)\| \leq C$ for $t \in [0, t_0]$, $\lambda \in \Lambda$ and $x \in \Omega$. Therefore, if $t, t' \in [0, t_0]$ are such that $t < t'$, then

$$\begin{aligned} \left\| \int_t^{t'} S_A(t' - s) F(\lambda, s, u(s; \lambda, x)) ds \right\| &\leq \int_t^{t'} Me^{\omega(t'-s)} \|F(\lambda, s, u(s; \lambda, x))\| ds \\ &\leq \int_t^{t'} Me^{|\omega|(t'-s)} c(s) (1 + \|u(s; \lambda, x)\|) ds = (t' - t) MKe^{|\omega|t_0} (1 + C). \end{aligned}$$

Taking $\delta := \varepsilon (MKe^{|\omega|t_0} (1 + C))^{-1}$ we obtain the assertion.

- (c) For any $\lambda \in \Lambda$ and $x \in \Omega$

$$\begin{aligned} \left\| \int_0^{t_0} S_A(t_0 - s) F(\lambda, s, u(s; \lambda, x)) ds \right\| &\leq \int_0^{t_0} Me^{\omega(t_0-s)} c(s) (1 + \|u(s; \lambda, x)\|) ds \\ &\leq \int_0^{t_0} KMe^{|\omega|t_0} (1 + \|u(s; \lambda, x)\|) ds \leq t_0 MKe^{|\omega|t_0} (1 + C) \end{aligned}$$

and $S(t_0)$ is bounded as claimed. \square

Proof of Theorem 2.1. Let $\Omega \subset X$ be a bounded set and let $t \in (0, +\infty)$. We shall prove first that the set $\Phi_t(\Lambda \times \Omega)$ is relatively compact. Let $\varepsilon > 0$. For $0 < t_0 < t$, $\lambda \in \Lambda$ and $x \in \Omega$

$$u(t; \lambda, x) = S_A(t)x + S_A(t - t_0) \left(\int_0^{t_0} S_A(t_0 - s) F(\lambda, s, u(s; \lambda, x)) ds \right) + \int_{t_0}^t S_A(t - s) F(\lambda, s, u(s; \lambda, x)) ds,$$

and, in consequence,

$$\begin{aligned} \{u(t; \lambda, x) \mid \lambda \in \Lambda, x \in \Omega\} &\subset S_A(t)\Omega + S_A(t - t_0)D_{t_0} \\ &\quad + \left\{ \int_{t_0}^t S_A(t - s) F(\lambda, s, u(s; \lambda, x)) ds \mid \lambda \in \Lambda, x \in \Omega \right\}, \end{aligned} \quad (2.11)$$

where

$$D_{t_0} := \left\{ \int_0^{t_0} S_A(t_0 - s) F(\lambda, s, u(s; \lambda, x)) ds \mid \lambda \in \Lambda, x \in \Omega \right\}.$$

Applying Lemma 2.3 (b), we infer that $t_0 \in (0, t)$ may be chosen so that

$$\left\| \int_{t_0}^t S_A(t - s) F(\lambda, s, u(s; \lambda, x)) ds \right\| \leq \varepsilon \quad \text{for } \lambda \in \Lambda, x \in \Omega. \quad (2.12)$$

From the point (c) of this lemma it follows that D_{t_0} is bounded. Combining (2.11) with (2.12) yields

$$\Phi_t(\Lambda \times \Omega) = \{u(t; \lambda, x) \mid \lambda \in \Lambda, x \in \Omega\} \subset V_\varepsilon + B(0, \varepsilon)$$

where $V_\varepsilon := S_A(t)\Omega + S_A(t - t_0)D_{t_0}$. This implies that V_ε is relatively compact, since $\{S_A(t)\}_{t \geq 0}$ is a compact semigroup and the sets Ω, D_{t_0} are bounded. On the other hand $\varepsilon > 0$ may be chosen arbitrary small and therefore the set $\Phi_t(\Lambda \times \Omega)$ is also relatively compact.

Let (λ_n) in Λ and (x_n) in X be sequences such that $\lambda_n \rightarrow \lambda_0 \in \Lambda$ and $x_n \rightarrow x_0 \in X$. We prove that $u(t; \lambda_n, x_n) \rightarrow u(t; \lambda_0, x_0)$ as $n \rightarrow +\infty$ uniformly on $[0, t_0]$ where $t_0 > 0$ is arbitrary. For every $n \geq 1$ write $u_n := u(\cdot; \lambda_n, x_n)$. We claim that (u_n) is an equicontinuous sequence of functions. Indeed, take $t \in [0, +\infty)$ and let $\varepsilon > 0$. If $h > 0$ then, by the integral formula,

$$u_n(t + h) - u_n(t) = S_A(h)u_n(t) - u_n(t) + \int_t^{t+h} S_A(t + h - s) F(\lambda_n, s, u_n(s)) ds. \quad (2.13)$$

Note that for every $t \in [0, +\infty)$ the set $\{u_n(t) \mid n \geq 1\}$ is relatively compact as proved earlier. For $t = 0$ it follows from the convergence of (x_n) , while for $t \in (0, +\infty)$ it is a consequence of the fact that the set $\Phi_t(\Lambda \times \{x_n \mid n \geq 1\})$ is relatively compact. From the continuity of semigroup there is $\delta_0 > 0$ such that

$$\|S_A(h)u_n(t) - u_n(t)\| \leq \varepsilon/2 \quad \text{for } h \in (0, \delta_0), n \geq 1. \quad (2.14)$$

By Lemma 2.3 (b) there is $\delta \in (0, \delta_0)$ such that for $h \in (0, \delta)$ and $n \geq 1$

$$\left\| \int_t^{t+h} S_A(t + h - s) F(\lambda_n, s, u_n(s)) ds \right\| \leq \varepsilon/2. \quad (2.15)$$

Combining (2.13), (2.14) and (2.15), for $h \in (0, \delta)$ we infer that

$$\|u_n(t + h) - u_n(t)\| \leq \|S_A(h)u_n(t) - u_n(t)\| + \left\| \int_t^{t+h} S_A(t + h - s) F(\lambda_n, s, u_n(s)) ds \right\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for every $n \geq 1$. We have thus proved that (u_n) is right-equicontinuous on $[0, +\infty)$. It remains to show that (u_n) is left-equicontinuous. To this end take $t \in (0, +\infty)$ and $\varepsilon > 0$. If h and δ are such that $0 < h < \delta < t$, then

$$\begin{aligned} \|u_n(t) - u_n(t - h)\| &\leq \|u_n(t) - S_A(\delta)u_n(t - \delta)\| + \|S_A(\delta)u_n(t - \delta) - S_A(\delta - h)u_n(t - \delta)\| \\ &\quad + \|S_A(\delta - h)u_n(t - \delta) - u_n(t - h)\|, \end{aligned} \quad (2.16)$$

and consequently, for any $n \geq 1$,

$$\begin{aligned} \|u_n(t) - u_n(t-h)\| &\leq \left\| \int_{t-\delta}^t S_A(t-s)F(\lambda_n, s, u_n(s)) ds \right\| + \|S_A(\delta)u_n(t-\delta) - S_A(\delta-h)u_n(t-\delta)\| \\ &\quad + \left\| \int_{t-\delta}^{t-h} S_A(t-h-s)F(\lambda_n, s, u_n(s)) ds \right\|. \end{aligned} \quad (2.17)$$

By Lemma 2.3 (b) there is $\delta \in (0, t)$ such that for every $t_1, t_2 \in [0, t]$ with $0 < t_1 - t_2 < \delta$, we have

$$\left\| \int_{t_1}^{t_2} S_A(t_2-s)F(\lambda_n, s, u_n(s)) ds \right\| \leq \varepsilon/3 \quad \text{for } n \geq 1. \quad (2.18)$$

Using again the relative compactness of $\{u_n(t) \mid n \geq 1\}$ where $t \in [0, +\infty)$ we can choose $\delta_1 \in (0, \delta)$ such that for every $h \in (0, \delta_1)$ and $n \geq 1$

$$\|S_A(\delta)u_n(t-\delta) - S_A(\delta-h)u_n(t-\delta)\| \leq \varepsilon/3. \quad (2.19)$$

Taking into account (2.17), (2.18), (2.19), for $h \in (0, \delta_1)$

$$\begin{aligned} \|u_n(t) - u_n(t-h)\| &\leq \left\| \int_{t-\delta}^t S_A(t-s)F(\lambda_n, s, u_n(s)) ds \right\| + \|S_A(\delta)u_n(t-\delta) - S_A(\delta-h)u_n(t-\delta)\| \\ &\quad + \left\| \int_{t-\delta}^{t-h} S_A(t-h-s)F(\lambda_n, s, u_n(s)) ds \right\| \leq \varepsilon, \end{aligned}$$

and finally the sequence (u_n) is left-equicontinuous on $(0, +\infty)$. Hence (u_n) is equicontinuous at every $t \in [0, +\infty)$ as claimed.

For every $n \geq 1$ write $w_n := u_n|_{[0, t_0]}$. We shall prove that $w_n \rightarrow w_0$ in $C([0, t_0], X)$ where $w_0 = u(\cdot; \lambda_0, x_0)|_{[0, t_0]}$. It is enough to show that every subsequence of (w_n) contains a subsequence convergent to w_0 . Let (w_{n_k}) be a subsequence of (w_n) . Since (w_{n_k}) is equicontinuous on $[0, t_0]$ and the set $\{w_{n_k}(s) \mid n \geq 1\} = \{u_{n_k}(s) \mid n \geq 1\}$ is relatively compact for any $s \in [0, t_0]$, by the Ascoli-Arzelà Theorem, we infer that (w_{n_k}) has a subsequence $(w_{n_{k_l}})$ such that $w_{n_{k_l}} \rightarrow w$ in $C([0, t_0], X)$ as $l \rightarrow +\infty$. For every $l \geq 1$ define a mapping $\phi_l : [0, t_0] \rightarrow X$ by

$$\phi_l(s) := S_A(t-s)F(\lambda_{n_{k_l}}, s, w_{n_{k_l}}(s)).$$

From the continuity of $\{S_A(t)\}_{t \geq 0}$ and F , we deduce that $\phi_l \rightarrow \phi$ in $C([0, t_0], X)$, where $\phi : [0, t_0] \rightarrow X$ is given by $\phi(s) = S_A(t-s)F(\lambda_0, s, w_0(s))$. It is clear that

$$w_{n_{k_l}}(t') = S_A(t')x_0 + \int_0^{t'} \phi_l(s) ds \quad \text{for } t' \in [0, t_0],$$

and therefore, passing to the limit with $l \rightarrow \infty$, we infer that for $t' \in [0, t_0]$

$$w_0(t') = S_A(t')x_0 + \int_0^{t'} \phi(s) ds = S_A(t')x_0 + \int_0^{t'} S_A(t'-s)F(\lambda_0, s, w_0(s)) ds.$$

By the uniqueness of mild solutions, $w_0(t) = w(t)$ for $t' \in [0, t_0]$ and we conclude that $w_{n_{k_l}} \rightarrow w_0 = u(\cdot; \lambda_0, x_0)$ as $l \rightarrow \infty$ and finally that $w_n \rightarrow w_0$ in $C([0, t_0], X)$. This completes the proof of point (a). \square

If linear operator $A : D(A) \rightarrow X$ is defined on a complex space X , then the point spectrum of A is the set $\sigma_p(A) := \{\lambda \in \mathbb{C} \mid \text{there exists } z \in X \setminus \{0\} \text{ such that } \lambda z - Az = 0\}$. For a linear operator A defined on a real space X , we consider its complex point spectrum in the following way (see [2] or [10]). By the complexification of X we mean a complex linear space $(X_{\mathbb{C}}, +, \cdot)$, where $X_{\mathbb{C}} := X \times X$, with the operations of addition $+: X_{\mathbb{C}} \times X_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ and multiplication by complex scalars $\cdot : \mathbb{C} \times X_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ given by

$$\begin{aligned} (x_1, y_1) + (x_2, y_2) &:= (x_1 + x_2, y_1 + y_2) \quad \text{for } (x_1, y_1), (x_2, y_2) \in X_{\mathbb{C}}, \quad \text{and} \\ (\alpha + \beta i) \cdot (x, y) &:= (\alpha x - \beta y, \alpha y + \beta x) \quad \text{for } \alpha + \beta i \in \mathbb{C}, (x, y) \in X_{\mathbb{C}}, \end{aligned}$$

respectively. For convenience, denote the elements (x, y) of $X_{\mathbb{C}}$ by $x + yi$. If X is a space with a norm $\|\cdot\|$, then the mapping $\|\cdot\|_{\mathbb{C}} : X_{\mathbb{C}} \rightarrow \mathbb{R}$ given by

$$\|x + yi\|_{\mathbb{C}} := \sup_{\theta \in [-\pi, \pi]} \|\sin \theta x + \cos \theta y\|$$

is a norm on $X_{\mathbb{C}}$, and $(X_{\mathbb{C}}, \|\cdot\|_{\mathbb{C}})$ is a Banach space, provided X is so. The complexification of A is a linear operator $A_{\mathbb{C}} : D(A_{\mathbb{C}}) \rightarrow X_{\mathbb{C}}$ given by

$$D(A_{\mathbb{C}}) := D(A) \times D(A) \quad \text{and} \quad A_{\mathbb{C}}(x + yi) := Ax + Ayi \quad \text{for } x + yi \in D(A_{\mathbb{C}}).$$

Now, one can define the complex point spectrum of A by $\sigma_p(A) := \sigma_p(A_{\mathbb{C}})$.

Remark 2.4. If $-A$ is a generator of a C_0 semigroup $\{S_A(t)\}_{t \geq 0}$, then it is easy to check that the family $\{S_A(t)_{\mathbb{C}}\}_{t \geq 0}$ of the complexified operators is a C_0 semigroup of bounded linear operators on $X_{\mathbb{C}}$ with the generator $-A_{\mathbb{C}}$.

In the following proposition we mention some spectral properties of C_0 semigroups:

Proposition 2.5. (See [15, Theorem 16.7.2].) If $-A$ is the generator of a C_0 semigroup $\{S_A(t)\}_{t \geq 0}$ of bounded linear operators on a complex Banach space X , then

$$\sigma_p(S_A(t)) \setminus \{0\} = e^{-\sigma_p(A)} \quad \text{for } t > 0.$$

Furthermore, if $\lambda \in \sigma_p(A)$ then for every $t > 0$

$$\text{Ker}(e^{-\lambda t} I - S_A(t)) = \overline{\text{span}} \left(\bigcup_{k \in \mathbb{Z}} \text{Ker}(\lambda_{k,t} I - A) \right) \quad (2.20)$$

where $\lambda_{k,t} := \lambda + (2k\pi/t)i$ for $k \in \mathbb{Z}$.

3. Averaging principle for equations at resonance

In this section we are interested in the periodic problems of the form

$$\begin{cases} \dot{u}(t) = -Au(t) + \varepsilon F(t, u(t)), & t > 0, \\ u(t) = u(t + T), & t \geq 0 \end{cases} \quad (3.21)$$

where $T > 0$ is a fixed period, $\varepsilon \in [0, 1]$ is a parameter, $A : D(A) \rightarrow X$ is a linear operator on a real Banach space X and $F : [0, +\infty) \times X \rightarrow X$ is a continuous mapping. Suppose that F satisfies (F1) and (F2) and $-A$ generates a compact C_0 semigroup $\{S_A(t)\}_{t \geq 0}$ such that

(A1) $\text{Ker } A = \text{Ker}(I - S_A(T)) \neq \{0\}$;

(A2) there is a closed subspace $M \subset X$, $M \neq \{0\}$ such that $X = \text{Ker } A \oplus M$ and $S_A(t)M \subset M$ for $t \geq 0$.

Remark 3.1. (a) If A is any linear operator such that $-A$ generates a C_0 semigroup $\{S_A(t)\}_{t \geq 0}$, then it is immediate that $\text{Ker } A \subset \text{Ker}(I - S_A(t))$ for $t \geq 0$.

(b) Condition (A1) can be characterized in terms of the point spectrum. Namely, (A1) is satisfied if and only if

$$\{(2k\pi/T)i \mid k \in \mathbb{Z}, k \neq 0\} \cap \sigma_p(A) = \emptyset. \quad (3.22)$$

To see this suppose first that (A1) holds. If $(2k\pi/T)i \in \sigma_p(A)$ for some $k \neq 0$, then there is $z = x + yi \in X_{\mathbb{C}} \setminus \{0\}$ such that

$$A_{\mathbb{C}} z = (2k\pi/T)zi. \quad (3.23)$$

We actually know that $-A_{\mathbb{C}}$ is a generator of the C_0 semigroup $\{S_{A_{\mathbb{C}}}(t)\}_{t \geq 0}$ with $S_{A_{\mathbb{C}}}(t) = S_A(t)_{\mathbb{C}}$ for $t \geq 0$. Therefore, by Proposition 2.5, we find that $z \in \text{Ker}(I - S_{A_{\mathbb{C}}}(T))$ and, in consequence,

$$S_A(T)x + S_A(T)yi = x + yi.$$

By (A1), we get $Ax = Ay = 0$ and finally $A_{\mathbb{C}} z = 0$, contrary to (3.23). Conversely, suppose that (3.22) is satisfied. Operator $A_{\mathbb{C}}$ as a generator of a C_0 semigroup is closed, and hence $\text{Ker } A_{\mathbb{C}}$ is a closed subspace of $X_{\mathbb{C}}$. On the other hand, by (2.20) and (3.22),

$$\text{Ker}(I - S_A(T)_{\mathbb{C}}) = \text{Ker}(I - S_{A_{\mathbb{C}}}(T)) = \text{cl } \text{Ker } A_{\mathbb{C}} = \text{Ker } A_{\mathbb{C}},$$

which implies that $\text{Ker}(I - S_A(T)) = \text{Ker } A$, i.e. (A1) is satisfied.

Since X is a Banach space and M, N are closed subspaces, there are projections $P : X \rightarrow X$ and $Q : X \rightarrow X$ such that $P^2 = P$, $Q^2 = Q$, $P + Q = I$ and $\text{Im } P = N$, $\text{Im } Q = M$. Let $\Phi_T^\varepsilon : X \rightarrow X$ be the translation along trajectories operator associated with

$$\dot{u}(t) = -Au(t) + \varepsilon F(t, u(t)), \quad t > 0$$

and let μ denote the sum of the algebraic multiplicities of eigenvalues of $S_A(T)$ lying in $(1, +\infty)$. The compactness of the semigroup $\{S_A(t)\}_{t \geq 0}$ implies that the nonzero real eigenvalues of $S_A(T)$ form a sequence which is either finite or converges to 0 and the algebraic multiplicity of each of them is finite. In both cases, only a finite number of eigenvalues is greater than 1 and hence μ is well defined.

We are ready to formulate the main result of this section:

Theorem 3.2. Let $g : N \rightarrow N$ be a mapping given by

$$g(x) := \int_0^T P F(s, x) ds \quad \text{for } x \in N$$

and let $U \subset N$ and $V \subset M$ with $0 \in V$ be open bounded sets. If $g(x) \neq 0$ for $x \in \partial_N U$, then there is $\varepsilon_0 \in (0, 1)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and $x \in \partial(U \oplus V)$, $\Phi_T^\varepsilon(x) \neq x$ and

$$\deg_{\text{LS}}(I - \Phi_T^\varepsilon, U \oplus V) = (-1)^{\mu + \dim N} \deg_B(g, U)$$

where \deg_{LS} and \deg_B stand for the Leray–Schauder and the Brouwer topological degree, respectively.

Proof. Throughout the proof, we write $W := U \oplus V$ and $\Lambda := [0, 1] \times [0, 1] \times \overline{W}$. For any $(\varepsilon, s, y) \in \Lambda$ consider the differential equation

$$\dot{u}(t) = -Au(t) + G(\varepsilon, s, y, t, u(t)), \quad t > 0 \quad (3.24)$$

where $G : \Lambda \times [0, +\infty) \times X \rightarrow X$ is defined by

$$G(\varepsilon, s, y, t, x) := \varepsilon P F(t, sx + (1-s)Py) + \varepsilon s Q F(t, x).$$

We check that G satisfies condition (F1). Indeed, fix $(\varepsilon, s, y) \in \Lambda$ and take $x_0 \in X$. If $s = 0$ then $G(\varepsilon, s, y, t, \cdot)$ is constant, hence we may suppose that $s \neq 0$. There are constants $L_0, L_1 > 0$ and neighborhoods $V_0, V_1 \subset X$ of points $sx_0 + (1-s)Py$ and x_0 , respectively, such that

$$\|F(t, x_1) - F(t, x_2)\| \leq L_0 \|x_1 - x_2\| \quad \text{for } x_1, x_2 \in V_0, t \in [0, +\infty)$$

and

$$\|F(t, x_1) - F(t, x_2)\| \leq L_1 \|x_1 - x_2\| \quad \text{for } x_1, x_2 \in V_1, t \in [0, +\infty).$$

Then $V' := \frac{1}{s}(V_0 - (1-s)Py) \cap V_1$ is open, $x_0 \in V'$ and, for any $x_1, x_2 \in V'$,

$$\begin{aligned} & \|G(\varepsilon, s, y, t, x_1) - G(\varepsilon, s, y, t, x_2)\| \\ & \leq \varepsilon \|P\| \|F(t, sx_1 + (1-s)Py) - F(t, sx_2 + (1-s)Py)\| + s\varepsilon \|Q\| \|F(t, x_1) - F(t, x_2)\| \\ & \leq \varepsilon L_0 \|P\| \|x_1 - x_2\| + s\varepsilon L_1 \|Q\| \|x_1 - x_2\| \leq (L_0 \|P\| + L_1 \|Q\|) \|x_1 - x_2\|, \end{aligned}$$

i.e. (F1) is satisfied. An easy computation shows that condition (F2) also holds true.

If $(\varepsilon, s, y) \in \Lambda$ and $x \in X$, then by $u(\cdot; \varepsilon, s, y, x) : [0, +\infty) \rightarrow X$ we denote unique mild solution of (3.24) starting at x . For $t \geq 0$, let $\Theta_t : \Lambda \times X \rightarrow X$ be the translation along trajectories operator given by

$$\Theta_t(\varepsilon, s, y, x) := u(t; \varepsilon, s, y, x) \quad \text{for } (\varepsilon, s, y) \in \Lambda, x \in X, t \in [0, +\infty).$$

For every $\varepsilon \in (0, 1)$ we define the mapping $M^\varepsilon : [0, 1] \times \overline{W} \rightarrow X$ by

$$M^\varepsilon(s, x) := \Theta_T(\varepsilon, s, x, x).$$

Clearly M^ε is completely continuous for every $\varepsilon \in (0, 1)$. Indeed, by Theorem 2.1 the operator Θ_T is completely continuous and, consequently, the set $\Theta_T(\Lambda \times \overline{W}) \subset X$ is relatively compact. Since

$$M^\varepsilon([0, 1] \times \overline{W}) = \Theta_T(\{\varepsilon\} \times [0, 1] \times \overline{W} \times \overline{W}) \subset \Theta_T(\Lambda \times \overline{W}),$$

the set $M^\varepsilon([0, 1] \times \overline{W})$ is relatively compact as well.

Now we claim that there is $\varepsilon_0 \in (0, 1)$ such that

$$M^\varepsilon(s, x) \neq x \quad \text{for } x \in \partial W, s \in [0, 1], \varepsilon \in (0, \varepsilon_0]. \quad (3.25)$$

Suppose to the contrary that there are sequences (ε_n) in $(0, 1)$, (s_n) in $[0, 1]$ and (x_n) in ∂W such that $\varepsilon_n \rightarrow 0$ and

$$\Theta_T(\varepsilon_n, s_n, x_n, x_n) = M^{\varepsilon_n}(s_n, x_n) = x_n \quad \text{for } n \geq 1. \quad (3.26)$$

We may assume that $s_n \rightarrow s_0$ with $s_0 \in [0, 1]$. By (3.26) and the boundedness of $(x_n) \subset \partial W$, the complete continuity of Θ_T implies that (x_n) has a convergent subsequence. Without loss of generality we may assume that $x_n \rightarrow x_0$ as $n \rightarrow +\infty$, for some $x_0 \in \partial W$. After passing to the limit in (3.26), by Theorem 2.1 (a), it follows that

$$\Theta_T(0, s_0, x_0, x_0) = x_0. \quad (3.27)$$

On the other hand

$$\Theta_t(0, s_0, x_0, x_0) = S_A(t)x_0 \quad \text{for } t \geq 0, \quad (3.28)$$

which together with (3.27) implies that $x_0 = S_A(T)x_0$. Condition (A1) yields $x_0 \in \text{Ker } A = N$ and hence $Qx_0 = 0$. Since $0 \in V$, and the equality

$$\partial(U \oplus V) = \partial_N U \oplus \text{cl}_M V \cup \text{cl}_N U \oplus \partial_M V$$

holds true, we infer that $x_0 \in \partial_N U$. By using of Remark 3.1 (a) and (3.28) we also find that

$$\Theta_t(0, s_0, x_0, x_0) = S_A(t)x_0 = x_0 \quad \text{for } t \geq 0. \quad (3.29)$$

For every $n \geq 1$, write $u_n := u(\cdot; \varepsilon_n, s_n, x_n, x_n)$ for brevity. As a consequence of (3.26)

$$\begin{aligned} x_n &= S_A(T)x_n + \varepsilon_n \int_0^T S_A(T-\tau)PF(\tau, s_n u_n(\tau) + (1-s_n)Px_n) d\tau \\ &\quad + \varepsilon_n s_n \int_0^T S_A(T-\tau)QF(\tau, u_n(\tau)) d\tau \quad \text{for } n \geq 1. \end{aligned} \quad (3.30)$$

The fact that the spaces $M, N \subset X$ are closed and $S_A(t)N \subset N$, $S_A(t)M \subset M$, for $t \geq 0$, leads to

$$\begin{aligned} \varepsilon_n \int_0^T S_A(T-\tau)PF(\tau, s_n u_n(\tau) + (1-s_n)Px_n) d\tau &\in N \quad \text{and} \\ \varepsilon_n s_n \int_0^T S_A(T-\tau)QF(\tau, u_n(\tau)) d\tau &\in M \quad \text{for } n \geq 1. \end{aligned} \quad (3.31)$$

Combining (3.30) with (3.31) gives

$$Px_n = S_A(T)Px_n + \varepsilon_n \int_0^T S_A(T-\tau)PF(\tau, s_n u_n(\tau) + (1-s_n)Px_n) d\tau \quad \text{for } n \geq 1,$$

and therefore

$$\int_0^T PF(\tau, s_n u_n(\tau) + (1-s_n)Px_n) d\tau = 0 \quad \text{for } n \geq 1, \quad (3.32)$$

since $Px_n \in \text{Ker } A = \text{Ker}(I - S_A(T))$ for $n \geq 1$. By Theorem 2.1 (a) and (3.29) the sequence (u_n) converges uniformly on $[0, T]$ to the constant mapping equal to x_0 , hence, passing to the limit in (3.32), we infer that

$$g(x_0) = \int_0^T PF(\tau, x_0) d\tau = 0.$$

This contradicts the assumption, since $x_0 \in \partial_N U$, and proves (3.25).

By the homotopy invariance of topological degree we have

$$\deg_{\text{LS}}(I - \Phi_T^\varepsilon, W) = \deg_{\text{LS}}(I - M^\varepsilon(1, \cdot), W) = \deg_{\text{LS}}(I - M^\varepsilon(0, \cdot), W) \quad (3.33)$$

for $\varepsilon \in (0, \varepsilon_0]$.

Let the mappings $\tilde{M}_1^\varepsilon : \bar{U} \rightarrow N$ and $\tilde{M}_2^\varepsilon : \bar{V} \rightarrow M$ be given by

$$\tilde{M}_1^\varepsilon(x_1) := x_1 + \varepsilon \int_0^T P F(s, x_1) ds \quad \text{for } x_1 \in \bar{U},$$

$$\tilde{M}_2^\varepsilon(x_2) := S_A(T)|_M x_2 \quad \text{for } x_2 \in \bar{V}$$

and let $\tilde{M}^\varepsilon : \bar{U} \times \bar{V} \rightarrow N \times M$ be their product

$$\tilde{M}^\varepsilon(x_1, x_2) := (\tilde{M}_1^\varepsilon(x_1), \tilde{M}_2^\varepsilon(x_2)) \quad \text{for } (x_1, x_2) \in \bar{U} \times \bar{V}.$$

For $\varepsilon \in (0, 1)$ and $x \in X$

$$M^\varepsilon(0, x) = S_A(T)x + \varepsilon \int_0^T S_A(T - \tau) P F(\tau, Px) d\tau = S_A(T)x + \varepsilon \int_0^T P F(\tau, Px) d\tau,$$

and therefore it is easily seen that the mappings $M^\varepsilon(0, \cdot)$ and \tilde{M}^ε are topologically conjugate. By the compactness of the C_0 semigroup $\{S_A(t) : M \rightarrow M\}_{t \geq 0}$ and the fact that $\text{Ker}(I - S_A(T)|_M) = 0$, we infer that the mapping

$$I - \tilde{M}_2^\varepsilon : M \rightarrow M$$

is a linear isomorphism. By use of the multiplication property of topological degree, for any $\varepsilon \in (0, 1)$,

$$\begin{aligned} \deg_{\text{LS}}(I - M^\varepsilon(0, \cdot), W) &= \deg_{\text{LS}}(I - \tilde{M}^\varepsilon, U \times V) \\ &= \deg_B(I - \tilde{M}_1^\varepsilon, U) \cdot \deg_{\text{LS}}(I - \tilde{M}_2^\varepsilon, V). \end{aligned}$$

Combining this with (3.33), we conclude that

$$\begin{aligned} \deg_{\text{LS}}(I - \Phi_T^\varepsilon, W) &= \deg_B(-\varepsilon g, U) \cdot \deg_{\text{LS}}(I - S_A(T)|_M, V) \\ &= (-1)^{\dim N} \deg_B(g, U) \cdot \deg_{\text{LS}}(I - S_A(T)|_M, V), \end{aligned}$$

for $\varepsilon \in (0, \varepsilon_0]$. If $\lambda \neq 1$ and $k \geq 1$ is an integer then, by (A1) and (A2),

$$\text{Ker}(\lambda I - S_A(T))|_M^k = \text{Ker}(\lambda I - S_A(T))^k.$$

Hence $\sigma_p(S_A(T)|_M) = \sigma_p(S_A(T)) \setminus \{1\}$ and the algebraic multiplicities of the corresponding eigenvalues are the same. Therefore, by the standard spectral properties of compact operators (see e.g. [11, Theorem 12.8.3]),

$$\deg_{\text{LS}}(I - S_A(T)|_M, V) = (-1)^\mu,$$

and finally

$$\deg_{\text{LS}}(I - \Phi_T^\varepsilon, W) = (-1)^{\mu + \dim N} \deg_B(g, U),$$

for every $\varepsilon \in (0, \varepsilon_0]$, which completes the proof. \square

An immediate consequence of Theorem 3.2 is the following

Corollary 3.3. *Let $U \subset N$ and $V \subset M$ with $0 \in V$ be open bounded sets such that $g(x) \neq 0$ for $x \in \partial_N U$. If $\deg_B(g, U) \neq 0$, then there is $\varepsilon_0 \in (0, 1)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ problem (3.21) admits a T -periodic mild solution.*

4. Periodic problems with the Landesman–Lazer type conditions

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open bounded set and let $X := L^2(\Omega)$. By $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ we denote the usual norm and scalar product on X , respectively. Assume that continuous mapping $f : [0, +\infty) \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following conditions:

(a) there is a constant $m > 0$ such that

$$|f(t, x, y)| \leq m \quad \text{for } t \in [0, +\infty), x \in \Omega, y \in \mathbb{R};$$

(b) there is a constant $L > 0$ such that for any $t \in [0, +\infty)$, $x \in \Omega$ and $y_1, y_2 \in \mathbb{R}$

$$|f(t, x, y_1) - f(t, x, y_2)| \leq L|y_1 - y_2|;$$

(c) $f(t, x, y) = f(t + T, x, y)$ for $t \in [0, +\infty)$, $x \in \Omega$ and $y \in \mathbb{R}$;

(d) there are continuous functions $f_+, f_- : [0, +\infty) \times \Omega \rightarrow \mathbb{R}$ such that

$$f_+(t, x) = \lim_{y \rightarrow +\infty} f(t, x, y) \quad \text{and} \quad f_-(t, x) = \lim_{y \rightarrow -\infty} f(t, x, y)$$

for $t \in [0, +\infty)$ and $x \in \Omega$.

Consider the following periodic differential problem

$$\begin{cases} \dot{u}(t) = -Au(t) + \lambda u(t) + F(t, u(t)), & t > 0, \\ u(t) = u(t + T), & t \geq 0 \end{cases} \quad (4.34)$$

where $A : D(A) \rightarrow X$ is a linear operator such that $-A$ generates a compact C_0 semigroup $\{S_A(t)\}_{t \geq 0}$ of bounded linear operators on X , λ is a real eigenvalue of A and $F : [0, +\infty) \times X \rightarrow X$ is a continuous mapping given by the formula

$$F(t, u)(x) := f(t, x, u(x)) \quad \text{for } t \in [0, +\infty), x \in \Omega.$$

Additionally, we suppose that

$$(A3) \quad \text{Ker}(A - \lambda I) = \text{Ker}(A^* - \lambda I) = \text{Ker}(I - e^{\lambda T} S_A(T)).$$

Recall that by assumptions (a) and (b), the mapping F is well defined, bounded, continuous and Lipschitz uniformly with respect to time. Therefore, the translations along trajectories operator $\Psi_t : X \rightarrow X$ associated with

$$\dot{u}(t) = -Au(t) + \lambda u(t) + F(t, u(t)), \quad t > 0$$

is well defined and completely continuous for $t > 0$, as a consequence of Theorem 2.1. Let $N_\lambda := \text{Ker}(\lambda I - A)$ and define $g : N_\lambda \rightarrow N_\lambda$ by

$$g(u) := \int_0^T P F(t, u) dt \quad \text{for } u \in N_\lambda,$$

where $P : X \rightarrow X$ is the orthogonal projection onto N_λ . Since $\{S_A(t)\}_{t \geq 0}$ is compact, A has compact resolvents and $\dim N_\lambda < \infty$. Furthermore note that, for any $u, z \in N_\lambda$,

$$\begin{aligned} \langle g(u), z \rangle &= \int_0^T \langle P F(t, u), z \rangle dt = \int_0^T \langle F(t, u), z \rangle dt \\ &= \int_0^T \int_\Omega f(t, x, u(x)) z(x) dx dt. \end{aligned} \quad (4.35)$$

We are ready to state the main result of this section:

Theorem 4.1. Suppose that $f : [0, +\infty) \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies one of the following Landesman–Lazer type conditions:

$$\int_0^T \int_{\{u>0\}} f_+(t, x) u(x) dx dt + \int_0^T \int_{\{u<0\}} f_-(t, x) u(x) dx dt > 0, \quad (4.36)$$

for any $u \in N_\lambda$ with $\|u\| = 1$, or

$$\int_0^T \int_{\{u>0\}} f_+(t, x) u(x) dx dt + \int_0^T \int_{\{u<0\}} f_-(t, x) u(x) dx dt < 0, \quad (4.37)$$

for any $u \in N_\lambda$ with $\|u\| = 1$. Then the problem (4.34) admits a T -periodic mild solution.

In the proof of preceding theorem, we use the following

Theorem 4.2. Let $f : [0, +\infty) \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy the following condition:

$$\int_0^T \int_{\{u>0\}} f_+(t, x) u(x) dx dt + \int_0^T \int_{\{u<0\}} f_-(t, x) u(x) dx dt \neq 0 \quad (4.38)$$

for every $u \in N_\lambda$ with $\|u\| = 1$. Then there is $R > 0$ such that $\Psi_T(u) \neq u$ for $u \in X \setminus B(0, R)$, $g(u) \neq 0$ for $u \in N_\lambda \setminus B(0, R)$ and

$$\deg_{LS}(I - \Psi_T, B(0, R)) = (-1)^{\mu(\lambda) + \dim N_\lambda} \deg_B(g, B(0, R) \cap N_\lambda) \quad (4.39)$$

where $\mu(\lambda)$ is the sum of the algebraic multiplicities of the eigenvalues of $e^{\lambda T} S_A(T) : X \rightarrow X$ lying in $(1, +\infty)$.

We shall use the following lemma:

Lemma 4.3. *If $f : [0, +\infty) \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies (4.38), then there is $R_0 > 0$ such that $g(u) \neq 0$ for $u \in N_\lambda$ with $\|u\| \geq R_0$.*

Proof. Suppose the assertion is false. Then there is a sequence $(u_n) \subset N_\lambda$ such that $g(u_n) = 0$ for $n \geq 1$ and $\|u_n\| \rightarrow +\infty$ as $n \rightarrow +\infty$. Define $z_n := u_n / \|u_n\|$ for $n \geq 1$. Since $(z_n) \subset N_\lambda$ and N_λ is a finite dimensional space, (z_n) is relatively compact. We can assume that there is $z_0 \in N_\lambda$ with $\|z_0\| = 1$ such that $z_n \rightarrow z_0$ as $n \rightarrow +\infty$. Additionally, we can suppose that $z_n(x) \rightarrow z_0(x)$ as $n \rightarrow +\infty$ for almost every $x \in \Omega$. Let

$$\Omega_+ := \{x \in \Omega \mid z_0(x) > 0\} \quad \text{and} \quad \Omega_- := \{x \in \Omega \mid z_0(x) < 0\}. \quad (4.40)$$

Then, by (4.35), we have

$$0 = \langle g(u_n), z_0 \rangle = \int_0^T \int_{\Omega} f(t, x, u_n(x)) z_0(x) dx dt, \quad \text{for } n \geq 1$$

and therefore

$$\int_0^T \int_{\Omega_+} f(t, x, z_n(x) \|u_n\|) z_0(x) dx dt + \int_0^T \int_{\Omega_-} f(t, x, z_n(x) \|u_n\|) z_0(x) dx dt = 0, \quad (4.41)$$

for $n \geq 1$. Note that, for fixed $t \in [0, T]$, the convergence $f(t, x, z_n(x) \|u_n\|) \rightarrow f_+(t, x)$ by $n \rightarrow +\infty$ occurs for almost every $x \in \Omega_+$. Since the domain Ω has finite measure, $z_0 \in L^2(\Omega) \subset L^1(\Omega)$. From the boundedness of f and the dominated convergence theorem, we infer that, for any $t \in [0, T]$,

$$\int_{\Omega_+} f(t, x, z_n(x) \|u_n\|) z_0(x) dx \rightarrow \int_{\Omega_+} f_+(t, x) z_0(x) dx \quad \text{as } n \rightarrow +\infty. \quad (4.42)$$

The function $\varphi_n^+ : [0, T] \rightarrow \mathbb{R}$ given by

$$\varphi_n^+(t) := \int_{\Omega_+} f(t, x, z_n(x) \|u_n\|) z_0(x) dx = \langle F(t, u_n), \max(z_0, 0) \rangle \quad \text{for } t \in [0, T]$$

is continuous and furthermore $|\varphi_n^+(t)| \leq m \|z_0\|_{L^1(\Omega)} < +\infty$ for $t \in [0, T]$. By use of (4.42) and the dominated convergence theorem, we deduce that

$$\int_0^T \int_{\Omega_+} f(t, x, z_n(x) \|u_n\|) z_0(x) dx dt \rightarrow \int_0^T \int_{\Omega_+} f_+(t, x) z_0(x) dx dt$$

as $n \rightarrow +\infty$. Proceeding in the same way, we also find that

$$\int_0^T \int_{\Omega_-} f(t, x, z_n(x) \|u_n\|) z_0(x) dx dt \rightarrow \int_0^T \int_{\Omega_-} f_-(t, x) z_0(x) dx dt$$

as $n \rightarrow +\infty$. In consequence, after passing to the limit in (4.41)

$$\int_0^T \int_{\Omega_+} f_+(t, x) z_0(x) dx dt + \int_0^T \int_{\Omega_-} f_-(t, x) z_0(x) dx dt = 0$$

for $z_0 \in N_\lambda$ with $\|z_0\| = 1$, contrary to (4.38), which completes the proof. \square

Proof of Theorem 4.2. Consider the following differential problem

$$\dot{u}(t) = -Au(t) + \lambda u(t) + \varepsilon F(t, u(t)), \quad t > 0$$

where ε is a parameter from $[0, 1]$ and let $\gamma_t : [0, 1] \times X \rightarrow X$ be the translations along trajectories operator for this equation. The previous lemma shows that there is $R_0 > 0$ such that $g(u) \neq 0$ for $u \in N_\lambda$ with $\|u\| \geq R_0$. We claim that there is $R_1 \geq R_0$ such that

$$\gamma_T(\varepsilon, u) \neq u \quad \text{for } \varepsilon \in (0, 1], \quad u \in X, \quad \|u\| \geq R_1. \quad (4.43)$$

Conversely, suppose that there are sequences (ε_n) in $(0, 1]$ and (u_n) in X such that

$$\gamma_T(\varepsilon_n, u_n) = u_n \quad \text{for } n \geq 1 \quad (4.44)$$

and $\|u_n\| \rightarrow +\infty$ as $n \rightarrow +\infty$. For every $n \geq 1$, set $w_n := w(\cdot; \varepsilon_n, u_n)$ where $w(\cdot; \varepsilon, u)$ is a mild solution of

$$\dot{w}(t) = -Aw(t) + \lambda w(t) + \varepsilon F(t, w(t))$$

starting at u . Then

$$w_n(t) = e^{\lambda t} S_A(t) u_n + \varepsilon_n \int_0^t e^{\lambda(t-s)} S_A(t-s) F(s, w_n(s)) ds \quad (4.45)$$

for $n \geq 1$ and $t \in [0, +\infty)$. Putting $t := T$ in the above equation, by (4.44), we infer that

$$z_n = e^{\lambda T} S_A(T) z_n + v_n(T), \quad (4.46)$$

with $z_n := u_n / \|u_n\|$ and

$$v_n(t) := \frac{\varepsilon_n}{\|u_n\|} \int_0^t e^{\lambda(t-s)} S_A(t-s) F(s, w_n(s)) ds \quad \text{for } n \geq 1, \quad t \in [0, +\infty).$$

Observe that, for any $t \in [0, T]$ and $n \geq 1$, we have

$$\|v_n(t)\| \leq \frac{1}{\|u_n\|} \int_0^t M e^{(\omega+\lambda)(t-s)} \|F(s, w_n(s))\| ds \leq M \nu(\Omega)^{1/2} M e^{T(|\omega|+|\lambda|)} / \|u_n\| \quad (4.47)$$

where the constants $M \geq 1$ and $\omega \in \mathbb{R}$ are such that $\|S_A(t)\| \leq M e^{\omega t}$ for $t \geq 0$ and ν stands for the Lebesgue measure. Hence

$$v_n(t) \rightarrow 0 \quad \text{for } t \in [0, T] \text{ as } n \rightarrow +\infty, \quad (4.48)$$

and, in particular, set $\{v_n(T)\}_{n \geq 1}$ is relatively compact. In view of (4.46)

$$\{z_n\}_{n \geq 1} \subset e^{\lambda T} S_A(T) (\{z_n\}_{n \geq 1}) + \{v_n(T)\}_{n \geq 1}, \quad (4.49)$$

and therefore, by the compactness of $\{S_A(t)\}_{t \geq 0}$ we see that $\{z_n\}_{n \geq 1}$ has a convergent subsequence. Without loss of generality we may assume that $z_n \rightarrow z_0$ as $n \rightarrow +\infty$ and $z_n(x) \rightarrow z_0(x)$ for almost every $x \in \Omega$, where $z_0 \in X$ is such that $\|z_0\| = 1$. Passing to the limit in (4.46), as $n \rightarrow +\infty$, and using (4.48), we find that $z_0 = e^{\lambda T} S_A(T) z_0$, hence that $z_0 \in \text{Ker}(I - e^{\lambda T} S_A(T))$ and finally, by condition (A3), that

$$z_0 \in \text{Ker}(\lambda I - A) = \text{Ker}(\lambda I - A^*). \quad (4.50)$$

Thus Remark 3.1 (a) leads to

$$z_0 \in \text{Ker}(I - e^{\lambda t} S_A(t)) \quad \text{for } t \geq 0. \quad (4.51)$$

From (4.45) we deduce that

$$\frac{1}{\|u_n\|} (w_n(t) - u_n) = e^{\lambda t} S_A(t) z_n - z_n + v_n(t) \quad \text{for } t \in [0, T],$$

which by (4.48) and (4.51) gives

$$\frac{1}{\|u_n\|} (w_n(t) - u_n) \rightarrow 0 \quad \text{for } t \in [0, T] \text{ as } n \rightarrow +\infty. \quad (4.52)$$

If we again take $t := T$ in (4.45) and act with the scalar product operation $\langle \cdot, z_0 \rangle$, we obtain

$$\langle u_n, z_0 \rangle = \langle e^{\lambda T} S_A(T) u_n, z_0 \rangle + \varepsilon_n \int_0^T e^{\lambda(T-s)} \langle S_A(T-s) F(s, w_n(s)), z_0 \rangle ds.$$

Since X is Hilbert space, by [17, Corollary 1.10.6], the family $\{S_A(t)^*\}_{t \geq 0}$ of the adjoint operators is a C_0 semigroup on X with the generator $-A^*$, i.e.

$$S_A(t)^* = S_{A^*}(t) \quad \text{for } t \geq 0. \quad (4.53)$$

Remark 3.1 (a) and (4.50) imply that $z_0 \in \text{Ker}(I - e^{\lambda t} S_{A^*}(t))$ for $t \geq 0$ and consequently, by (4.53), $z_0 \in \text{Ker}(I - e^{\lambda t} S_A(t)^*)$ for $t \geq 0$. Thus

$$\begin{aligned} \langle u_n, z_0 \rangle &= \langle u_n, e^{\lambda T} S_A(T)^* z_0 \rangle + \varepsilon_n \int_0^T e^{\lambda(T-s)} \langle F(s, w_n(s)), S_A(T-s)^* z_0 \rangle ds \\ &= \langle u_n, z_0 \rangle + \varepsilon_n \int_0^T \langle F(s, w_n(s)), z_0 \rangle ds, \end{aligned}$$

and therefore

$$\int_0^T \langle F(s, w_n(s)), z_0 \rangle ds = 0 \quad \text{for } n \geq 1.$$

We have further

$$\begin{aligned} 0 &= \int_0^T \int_{\Omega} f(s, x, w_n(s)(x)) z_0(x) dx ds \\ &= \int_0^T \int_{\Omega_+} f(s, x, w_n(s)(x)) z_0(x) dx ds + \int_0^T \int_{\Omega_-} f(s, x, w_n(s)(x)) z_0(x) dx ds, \end{aligned} \quad (4.54)$$

where the sets Ω_+ and Ω_- are given by (4.40). Given $s \in [0, T]$, we claim that

$$\varphi_n^+(s) := \int_{\Omega_+} f(s, x, w_n(s)(x)) z_0(x) dx \rightarrow \int_{\Omega_+} f_+(s, x) z_0(x) dx \quad (4.55)$$

and

$$\varphi_n^-(s) := \int_{\Omega_-} f(s, x, w_n(s)(x)) z_0(x) dx \rightarrow \int_{\Omega_-} f_-(s, x) z_0(x) dx \quad (4.56)$$

as $n \rightarrow \infty$. Since the proofs of (4.55) and (4.56) are analogous, we consider only the former limit. We show that every sequence (n_k) of natural numbers has a subsequence (n_{k_l}) such that

$$\int_{\Omega_+} f(s, x, (h_{n_{k_l}}(s, x) + z_{n_{k_l}}(x)) \|u_{n_{k_l}}\|) z_0(x) dx \rightarrow \int_{\Omega_+} f_+(s, x) z_0(x) dx \quad (4.57)$$

as $n \rightarrow +\infty$ with

$$h_n(s, x) := (w_n(s)(x) - u_n(x)) / \|u_n\| \quad \text{for } x \in \Omega, \quad n \geq 1.$$

Due to (4.52), one can choose a subsequence $(h_{n_{k_l}}(s, \cdot))$ of $(h_{n_k}(s, \cdot))$ such that $h_{n_{k_l}}(s, x) \rightarrow 0$ for almost every $x \in \Omega$. Hence

$$h_{n_{k_l}}(s, x) + z_{n_{k_l}}(x) \rightarrow z_0(x) > 0 \quad \text{as } n \rightarrow +\infty \quad (4.58)$$

for almost every $x \in \Omega_+$ and consequently

$$f(s, x, (h_{n_{k_l}}(s, x) + z_{n_{k_l}}(x)) \|u_{n_{k_l}}\|) \rightarrow f_+(s, x) \quad \text{as } n \rightarrow +\infty \quad (4.59)$$

for almost every $x \in \Omega_+$. Since $z_0 \in L^2(\Omega) \subset L^1(\Omega)$ and f is bounded, from the Lebesgue dominated convergence theorem, we have the convergence (4.57) and hence (4.55). Further, for any $s \in [0, T]$ and $n \geq 1$, one has

$$|\varphi_n^+(s)| \leq \int_{\Omega_+} |f(s, x, w_n(s)(x)) z_0(x)| dx \leq m \int_{\Omega_+} |z_0(x)| dx \leq m \|z_0\|_{L^1(\Omega)} \quad (4.60)$$

and similarly

$$|\varphi_n^-(s)| \leq m \|z_0\|_{L^1(\Omega)} \quad \text{for } t \in [0, T] \text{ and } n \geq 1. \quad (4.61)$$

Since

$$\varphi_n^+(s) = \langle F(s, w_n(s)), \max(z_0, 0) \rangle \quad \text{and} \quad \varphi_n^-(s) = \langle F(s, w_n(s)), \min(z_0, 0) \rangle$$

for $s \in [0, T]$ and $n \geq 1$, functions φ_n^+ and φ_n^- are continuous on $[0, T]$. Using (4.55), (4.56), (4.60), (4.61) and the dominated convergence theorem, after passing to the limit in (4.54), we infer that

$$\int_0^T \int_{\Omega_+} f_+(s, x) z_0(x) dx ds + \int_0^T \int_{\Omega_-} f_-(s, x) z_0(x) dx ds = 0, \quad (4.62)$$

which contradicts (4.38), since $z_0 \in N_\lambda$ and $\|z_0\| = 1$ and, in consequence, proves (4.43).

Let $R := R_1$. By the homotopy invariance of topological degree, for any $\varepsilon \in (0, 1]$, we have

$$\begin{aligned} \deg_{LS}(I - \Psi_T, B(0, R)) &= \deg_{LS}(I - \gamma_T(1, \cdot), B(0, R)) \\ &= \deg_{LS}(I - \gamma_T(\varepsilon, \cdot), B(0, R)). \end{aligned} \quad (4.63)$$

Since A has compact resolvents $\text{Ker}(A^* - \lambda I)^\perp = \text{Im}(A - \lambda I)$ and therefore, by (A3), X admits the direct sum decomposition

$$X = N_\lambda \oplus \text{Im}(A - \lambda I).$$

Clearly the range and kernel of A are invariant under $S_A(t)$ for $t \geq 0$, hence putting $M := \text{Im}(\lambda I - A)$, condition (A2) is satisfied for $A - \lambda I$. Moreover $R \geq R_0$ and therefore, we also have that $g(u) \neq 0$ for $u \in N_\lambda$ with $\|u\| \geq R$. Let $U := B(0, R) \cap N_\lambda$ and $V := B(0, R) \cap M$. Then $g(u) \neq 0$ for $u \in \partial_{N_\lambda} U$ and clearly

$$B(0, R) \subset U \oplus V. \quad (4.64)$$

Therefore, by Theorem 3.2, there is $\varepsilon_0 \in (0, 1)$ such that, for any $\varepsilon \in (0, \varepsilon_0]$ and $u \in \partial(U \oplus V)$, $\gamma_T(\varepsilon, u) \neq u$ and

$$\deg_{LS}(I - \gamma_T(\varepsilon, \cdot), U \oplus V) = (-1)^{\mu(\lambda) + \dim N_\lambda} \deg_B(g, U), \quad (4.65)$$

where $\mu(\lambda)$ is the sum of algebraic multiplicities of eigenvalues of $S_{A-\lambda I}(T)$ in $(1, +\infty)$. In view of (4.64) and the fact that $R = R_1$ satisfies (4.43), we infer that

$$\{u \in U \oplus V \mid \gamma_T(\varepsilon_0, u) = u\} \subset B(0, R)$$

and, by the excision property,

$$\deg_{LS}(I - \gamma_T(\varepsilon_0, \cdot), U \oplus V) = \deg_{LS}(I - \gamma_T(\varepsilon_0, \cdot), B(0, R)). \quad (4.66)$$

Combining (4.65) with (4.66) yields

$$\deg_{LS}(I - \gamma_T(\varepsilon_0, \cdot), B(0, R)) = (-1)^{\mu(\lambda) + \dim N_\lambda} \deg_B(g, U), \quad (4.67)$$

which together with (4.63) implies

$$\deg_{LS}(I - \Psi_T, B(0, R)) = (-1)^{\mu(\lambda) + \dim N_\lambda} \deg_B(g, U) \quad (4.68)$$

and the proof is complete. \square

The following proposition allows us to determine the Brouwer degree of the mapping g .

Proposition 4.4.

(i) If condition (4.36) holds then there is $R_0 > 0$ such that $g(u) \neq 0$ for $u \in N_\lambda$ with $\|u\| \geq R_0$ and

$$\deg_B(g, B(0, R)) = 1 \quad \text{for } R \geq R_0.$$

(ii) If condition (4.37) holds then there is $R_0 > 0$ such that $g(u) \neq 0$ for $u \in N_\lambda$ with $\|u\| \geq R_0$ and

$$\deg_B(g, B(0, R)) = (-1)^{\dim N_\lambda} \quad \text{for } R \geq R_0.$$

Proof. (i) We begin by proving that there exists $R_0 > 0$ such that

$$\langle g(u), u \rangle > 0 \quad \text{for } u \in N_\lambda, \quad \|u\| \geq R_0. \quad (4.69)$$

Arguing by contradiction, suppose that there is a sequence $(u_n) \subset N_\lambda$ such that $\|u_n\| \rightarrow +\infty$ as $n \rightarrow +\infty$ and $\langle g(u_n), u_n \rangle \leq 0$, for $n \geq 1$. For every $n \geq 1$, write $z_n := u_n / \|u_n\|$. Since (z_n) is bounded and contained in the finite dimensional space N_λ , it contains a convergent subsequence. Without loss of generality we may assume that there is $z_0 \in N_\lambda$ with $\|z_0\| = 1$ such that $z_n \rightarrow z_0$ as $n \rightarrow +\infty$ and $z_n(x) \rightarrow z_0(x)$ as $n \rightarrow +\infty$ for almost every $x \in \Omega$. Recalling the notational convention (4.40), we have

$$\begin{aligned} 0 &\geq \langle g(u_n), z_n \rangle = \langle g(u_n), z_n - z_0 \rangle + \langle g(u_n), z_0 \rangle \\ &= \int_0^T \int_\Omega f(t, x, u_n(x)) z_0(x) dx dt + \langle g(u_n), z_n - z_0 \rangle \\ &= \int_0^T \int_{\Omega_+} f(t, x, z_n(x) \|u_n\|) z_0(x) dx dt + \int_0^T \int_{\Omega_-} f(t, x, z_n(x) \|u_n\|) z_0(x) dx dt + \langle g(u_n), z_n - z_0 \rangle. \end{aligned} \quad (4.70)$$

On the other hand, if we fix $t \in [0, T]$, then, by the condition (d), we have

$$f(t, x, z_n(x) \|u_n\|) \rightarrow f_+(t, x) \quad \text{as } n \rightarrow +\infty \quad (4.71)$$

for almost every $x \in \Omega_+$. Since f is assumed to be bounded and $z_0 \in L^1(\Omega)$, by the dominated convergence theorem, (4.71) shows that

$$\int_{\Omega_+} f(t, x, z_n(x) \|u_n\|) z_0(x) dx \rightarrow \int_{\Omega_+} f_+(t, x) z_0(x) dx \quad (4.72)$$

as $n \rightarrow \infty$. Let $\varphi_n^+ : [0, T] \rightarrow \mathbb{R}$ be given by

$$\varphi_n^+(t) := \int_{\Omega_+} f(t, x, z_n(x) \|u_n\|) z_0(x) dx = \langle F(t, u_n), \max(z_0, 0) \rangle$$

for $t \in [0, T]$. The function φ_n^+ is evidently continuous and $|\varphi_n^+(t)| \leq m \|z_0\|_{L^1(\Omega)}$ for $t \in [0, T]$. Applying (4.72) and the dominated convergence theorem, we find that

$$\int_0^T \int_{\Omega_+} f(t, x, z_n(x) \|u_n\|) z_0(x) dx dt \rightarrow \int_0^T \int_{\Omega_+} f_+(t, x) z_0(x) dx dt, \quad (4.73)$$

as $n \rightarrow +\infty$. Proceeding in the same way, we infer that

$$\int_0^T \int_{\Omega_-} f(t, x, z_n(x) \|u_n\|) z_0(x) dx dt \rightarrow \int_0^T \int_{\Omega_-} f_-(t, x) z_0(x) dx dt, \quad (4.74)$$

as $n \rightarrow +\infty$. Since the sequence $(g(u_n))$ is bounded, we see that

$$|\langle g(u_n), z_n - z_0 \rangle| \leq \|g(u_n)\| \|z_n - z_0\| \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \quad (4.75)$$

By (4.73), (4.74), (4.75), letting $n \rightarrow +\infty$ in (4.70), we assert that

$$\int_0^T \int_{\Omega_+} f_+(t, x) z_0(x) dx dt + \int_0^T \int_{\Omega_-} f_-(t, x) z_0(x) dx dt \leq 0,$$

contrary to (4.36).

Now, for any $R > R_0$, the mapping $H : [0, 1] \times N_\lambda \rightarrow N_\lambda$ given by

$$H(s, u) := sg(u) + (1 - s)u \quad \text{for } u \in N_\lambda,$$

has no zeros for $s \in [0, 1]$ and $u \in N_\lambda$ with $\|u\| = R$. If it were not true, then there would be $H(s, u) = 0$, for some $s \in [0, 1]$ and $u \in N_\lambda$ with $\|u\| = R$, and in consequence,

$$0 = \langle H(s, u), u \rangle = s \langle g(u), u \rangle + (1 - s) \langle u, u \rangle.$$

If $s = 0$ then $0 = \|u\|^2 = R^2$, which is impossible. If $s \in (0, 1]$, then $0 \geq \langle g(u), u \rangle$, which contradicts (4.69). By the homotopy invariance of the topological degree

$$\begin{aligned} \deg_B(g, B(0, R)) &= \deg_B(H(1, \cdot), B(0, R)) = \deg_B(H(0, \cdot), B(0, R)) \\ &= \deg_B(I, B(0, R)) = 1, \end{aligned}$$

and the proof of (i) is complete.

(ii) Proceeding by analogy to (i), we obtain the existence of $R_0 > 0$ such that

$$\langle g(u), u \rangle < 0 \quad \text{for } \|u\| \geq R_0. \quad (4.76)$$

This implies, that for every $R > R_0$, the homotopy $H : [0, 1] \times N_\lambda \rightarrow N_\lambda$ given by

$$H(s, u) := sg(u) - (1 - s)u \quad \text{for } u \in N_\lambda$$

is such that $H(s, u) \neq 0$ for $s \in [0, 1]$ and $u \in N_\lambda$ with $\|u\| = R$. Indeed, if $H(s, u) = 0$ for some $s \in [0, 1]$ and $u \in N_\lambda$ with $\|u\| = R$, then

$$0 = \langle H(s, u), u \rangle = s \langle g(u), u \rangle - (1 - s) \langle u, u \rangle.$$

Hence, if $s \in (0, 1]$, then $\langle g(u), u \rangle \geq 0$, contrary to (4.76). If $s = 0$, then $R^2 = \|u\|^2 = 0$, and again a contradiction. In consequence, by the homotopy invariance,

$$\deg_B(g, B(0, R)) = \deg_B(-I, B(0, R)) = (-1)^{\dim N_\lambda},$$

as desired. \square

Proof Theorem 4.1. Theorem 4.2 asserts that there is $R > 0$ such that $\Psi_T(u) \neq u$ for $u \in X \setminus B(0, R)$, $g(u) \neq 0$ for $u \in N_\lambda \setminus B(0, R)$ and

$$\deg_{LS}(I - \Psi_T, B(0, R)) = (-1)^{\mu(\lambda) + \dim N_\lambda} \deg_B(g, B(0, R) \cap N_\lambda). \quad (4.77)$$

In view of Proposition 4.4, we obtain the existence of $R_0 > R$ such that either $\deg(g, B(0, R_0) \cap N_\lambda) = 1$, when (4.36) is satisfied, or $\deg(g, B(0, R_0) \cap N_\lambda) = (-1)^{\dim N_\lambda}$, in the case of condition (4.37). By the inclusion $\{u \in B(0, R_0) \cap N_\lambda \mid g(u) = 0\} \subset B(0, R) \cap N_\lambda$, we see that

$$\deg(g, B(0, R) \cap N_\lambda) = \deg(g, B(0, R_0) \cap N_\lambda) = \pm 1$$

and, by (4.77),

$$\deg_{LS}(I - \Psi_T, B(0, R)) = (-1)^{\mu(\lambda) + \dim N_\lambda} \deg_B(g, B(0, R) \cap N_\lambda) = \pm 1.$$

Thus, by the existence property, we find that there is a fixed point of Ψ_T and in consequence a T -periodic mild solution of (4.34). \square

In the particular case when the linear operator A is self-adjoint and $-A$ is a generator of a compact C_0 semigroup $\{S_A(t)\}_{t \geq 0}$ of bounded linear operators on X , the spectrum $\sigma(A)$ is real and consists of eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_k < \dots$ which form a sequence convergent to infinity. By Proposition 2.5, for every $t > 0$, $\{e^{-\lambda_k t}\}_{k \geq 1}$ is the sequence of nonzero eigenvalues of $S_A(t)$ and

$$\text{Ker}(\lambda_k I - A) = \text{Ker}(e^{-\lambda_k t} I - S_A(t)) \quad \text{for } k \geq 1. \quad (4.78)$$

In consequence, we see that (A3) holds.

Corollary 4.5. Let A be a self-adjoint operator such that $-A$ is a generator of a compact C_0 semigroup $\{S_A(t)\}_{t \geq 0}$ and let $f : [0, +\infty) \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy the Landesman–Lazer type condition (4.38). If $\lambda = \lambda_k$ for some $k \geq 1$, then there is $R > 0$ such that $\Psi_T(u) \neq u$ for $u \in X \setminus B(0, R)$, $g(u) \neq 0$ for $u \in N_{\lambda_k} \setminus B(0, R)$ and

$$\deg_{LS}(I - \Psi_T, B(0, R)) = (-1)^{d_k} \deg_B(g, B(0, R) \cap N_{\lambda_k}), \quad (4.79)$$

where $d_k := \sum_{i=1}^{k-1} \dim \text{Ker}(\lambda_i I - A)$ for $k \geq 1$. In particular, if either condition (4.36) or (4.37) is satisfied then (4.34) has mild solution.

Proof. To see (4.79), it is enough to check that $d_k = \mu(\lambda_k) + \dim N_{\lambda_k}$ for $k \geq 1$. Since

$$e^{(\lambda_k - \lambda_1)T} > e^{(\lambda_k - \lambda_2)T} > \dots > e^{(\lambda_k - \lambda_{k-1})T}$$

are eigenvalues of $e^{\lambda_k T} S_A(T)$ which are greater than 1, for $k = 1$ it is evident that $\mu(\lambda_k) = 0$ and $d_1 = \mu(\lambda_k) + \dim N_{\lambda_k}$. The operator $S_A(T)$ is also self-adjoint and therefore the geometric and the algebraic multiplicity of each eigenvalue coincide. Hence

$$\mu(\lambda_k) = \sum_{i=1}^{k-1} \dim \text{Ker}(e^{-\lambda_i T} I - S_A(T)) \quad \text{for } k \geq 2. \quad (4.80)$$

From (4.78) and (4.80), we deduce that

$$\mu(\lambda_k) = \sum_{i=1}^{k-1} \dim \text{Ker}(\lambda_i I - A) = d_k - \dim N_{\lambda_k}$$

and finally that $d_k = \mu(\lambda_k) + \dim N_{\lambda_k}$ for every $k \geq 1$, as desired. The formula (4.79) together with Proposition 4.4 leads to existence of mild solution of (4.34) provided either condition (4.36) or (4.37) is satisfied. \square

5. Applications

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open bounded connected set with C^1 boundary. We recall that $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ denote, similarly as before, the norm and the scalar product on $X = L^2(\Omega)$, respectively. For $u \in H^1(\Omega)$, we will denote by $D_k u$, the k -th weak derivative of u .

5.1. Laplacian with the Neumann boundary conditions

We begin with the T -periodic parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + \varepsilon f(t, x, u) & \text{in } (0, +\infty) \times \Omega, \\ \frac{\partial u}{\partial n}(t, x) = 0 & \text{on } [0, +\infty) \times \partial\Omega, \\ u(t, x) = u(t + T, x) & \text{in } [0, +\infty) \times \Omega, \end{cases} \quad (5.81)$$

where $\varepsilon \in [0, 1]$ is a parameter and $f : [0, +\infty) \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous mapping which is required to satisfy conditions (a), (b) and (c) from the previous section. We put (5.81) into an abstract setting. To this end let $A : D(A) \rightarrow X$ be a linear operator such that $-A$ is the Laplacian with the Neumann boundary conditions, i.e.

$$D(A) := \left\{ u \in H^1(\Omega) \mid \text{there is } g \in L^2(\Omega) \text{ such that } \int_{\Omega} \nabla u \nabla h \, dx = \int_{\Omega} g h \, dx \text{ for } h \in H^1(\Omega) \right\},$$

$$Au := g, \quad \text{where } g \text{ is as above,}$$

and define $F : [0, +\infty) \times X \rightarrow X$ to be a mapping given by the formula

$$F(t, u)(x) := f(t, x, u(x)) \quad \text{for } t \in [0, +\infty), \, x \in \Omega. \quad (5.82)$$

Then by the assumptions (a) and (b), it is well defined, continuous, bounded and Lipschitz uniformly with respect to time. Problem (5.81) may be considered in the abstract form

$$\begin{cases} \dot{u}(t) = -Au(t) + \varepsilon F(t, u(t)), & t > 0, \\ u(t) = u(t + T), & t \geq 0 \end{cases} \quad (5.83)$$

where $\varepsilon \in [0, 1]$ is a parameter. Solutions of (5.81) will be understood as mild solutions of (5.83).

Theorem 5.1. Let $g_0 : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$g_0(y) := \int_0^T \int_{\Omega} f(t, x, y) \, dx \, dt \quad \text{for } y \in \mathbb{R}.$$

If real numbers a and b are such that $a < b$ and $g_0(a) \cdot g_0(b) < 0$, then there is $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$, the problem (5.81) admits a solution.

Proof. Since the spectrum of A is real, condition (A1) is satisfied as a consequence of Remark 3.1. It is known that $-A$ generates a compact C_0 semigroup on X and $N := \text{Ker } A$ is a one-dimensional space. Furthermore, if we take $M := \text{Im } A$, then $M = N^\perp$ and hence A satisfies also condition (A2). Let $P : X \rightarrow X$ be the orthogonal projection onto N given by

$$P(u) := \frac{1}{v(\Omega)}(u, e) \cdot e \quad \text{for } u \in X$$

where $e \in L^2(\Omega)$ represents the constant equal to 1 function and ν stands for the Lebesgue measure. Set $U := \{s \cdot e \mid s \in (a, b)\}$, $V := \{u \in N^\perp \mid \|u\| < 1\}$ and let $g: N \rightarrow N$ be defined by

$$g(u) := \int_0^T PF(t, u) dt \quad \text{for } u \in N.$$

Then

$$g_0(y) = \nu(\Omega) \cdot K^{-1}(g(K(y))) \quad \text{for } y \in \mathbb{R},$$

where $K: \mathbb{R} \rightarrow N$ is the linear homeomorphism given by $K(y) := y \cdot e$. Since $g_0(a) \cdot g_0(b) < 0$, we have $\deg_B(g, U) = \deg_B(g_0, (a, b)) \neq 0$ and hence, by Corollary 3.3, there is $\varepsilon_0 \in (0, 1)$ such that, for $\varepsilon \in (0, \varepsilon_0]$, problem (5.81) admits a solution as desired. \square

5.2. Differential operator with the Dirichlet boundary conditions

Suppose that $a^{ij} = a^{ji} \in C^1(\overline{\Omega})$ for $1 \leq i, j \leq n$ and let $\theta > 0$ be such that

$$a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \quad \text{for } \xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n, \quad x \in \Omega.$$

We assume that $A: D(A) \rightarrow X$ is a linear operator given by the formula

$$D(A) := \left\{ u \in H_0^1(\Omega) \mid \text{there is } g \in L^2(\Omega) \text{ such that } \int_{\Omega} a^{ij}(x) D_i u D_j h dx = \int_{\Omega} gh dx \text{ for } h \in H_0^1(\Omega) \right\},$$

$$Au := g, \quad \text{where } g \text{ is as above.}$$

It is well known that $-A$ is self-adjoint and generates a compact C_0 semigroup on $X = L^2(\Omega)$. Let $\lambda_1 < \lambda_2 < \dots < \lambda_k < \dots$ be the sequence of distinct eigenvalues of A . We are concerned with a periodic parabolic problem of the form

$$\begin{cases} u_t = D_i(a^{ij} D_j u) + \lambda_k u + f(t, x, u) & \text{in } (0, +\infty) \times \Omega, \\ u(t, x) = 0 & \text{on } [0, +\infty) \times \partial\Omega, \\ u(t, x) = u(t + T, x) & \text{in } [0, +\infty) \times \Omega, \end{cases} \quad (5.84)$$

where λ_k is k -th eigenvalue of A and $f: [0, +\infty) \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is as above. We write problem (5.84) in the abstract form

$$\begin{cases} \dot{u}(t) = -Au(t) + \lambda_k u(t) + F(t, u(t)), & t > 0, \\ u(t) = u(t + T), & t \geq 0, \end{cases}$$

where $F: [0, +\infty) \times X \rightarrow X$ is given by the formula (5.82). An immediate consequence of Corollary 4.5 is the following

Theorem 5.2. Suppose that $f: [0, +\infty) \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is such that:

$$\int_0^T \int_{\{u>0\}} f_+(t, x) u(x) dx dt + \int_0^T \int_{\{u<0\}} f_-(t, x) u(x) dx dt > 0,$$

for any $u \in \text{Ker } A$ with $\|u\| = 1$, or

$$\int_0^T \int_{\{u>0\}} f_+(t, x) u(x) dx dt + \int_0^T \int_{\{u<0\}} f_-(t, x) u(x) dx dt < 0,$$

for any $u \in \text{Ker } A$ with $\|u\| = 1$. Then the problem (5.84) admits a T -periodic mild solution.

Acknowledgments

The author wishes to thank Prof. W. Kryszewski and Dr. A. Ćwiszewski for helpful comments and suggestions, which raised the quality of this work.

References

- [1] A. Ambrosetti, G. Mancini, Existence and multiplicity results for nonlinear elliptic problems with linear part at resonance. The case of the simple eigenvalue, J. Differential Equations 28 (1978) 220–245.
- [2] H. Amann, Linear and Quasilinear Parabolic Problems, Birkhäuser, 1995.
- [3] H. Brezis, L. Nirenberg, Characterizations of the ranges of some nonlinear operators and applications to boundary value problems, Ann. Sc. Norm. Super. Pisa 5 (1978) 225–326.

- [4] L. Cesari, R. Kannan, Periodic solutions of nonlinear wave equations with damping, *Rend. Circ. Mat. Palermo* (2) 31 (1982) 421–432.
- [5] A. Ćwiszewski, Topological degree methods for perturbations of operators generating compact C_0 semigroups, *J. Differential Equations* 220 (2006) 434–477.
- [6] A. Ćwiszewski, Degree theory for perturbations of m -accretive operators generating compact semigroups with constraints, *J. Evol. Equ.* 7 (2007) 1–33.
- [7] A. Ćwiszewski, Periodic solutions of damped hyperbolic equations at resonance: a translation along trajectories approach, *Differential Integral Equations* 24 (7–8) (2011) 767–786.
- [8] A. Ćwiszewski, P. Kokocki, Krasnosel'skii type formula and translation along trajectories method for evolution equations, *Discrete Contin. Dyn. Syst.* 22 (2008) 605–628.
- [9] A. Ćwiszewski, P. Kokocki, Periodic solutions of nonlinear hyperbolic evolution systems, *J. Evol. Equ.* 10 (3) (2010) 677–710.
- [10] D. Daners, P.K. Medina, *Abstract Evolution Equations, Periodic Problems and Applications*, Pitman Res. Notes Math. Ser., Longman Scientific & Technical, 1992.
- [11] J. Dugundji, A. Granas, *Fixed Point Theory*, Springer-Verlag, 2004.
- [12] S. Fucik, J. Mawhin, Generalized periodic solutions of nonlinear telegraph equation, *Nonlinear Anal.* 2 (1978) 609–617.
- [13] M. Furi, M.P. Pera, Global bifurcation of fixed points and the Poincaré translation operator on manifolds, *Ann. Mat. Pura Appl.* (4) CLXXIII (1997) 313–331.
- [14] P. Hess, Nonlinear perturbations of linear elliptic and parabolic problems at resonance: existence of multiple solutions, *Ann. Sc. Norm. Super. Pisa* 5 (1978) 527–537.
- [15] E. Hille, R. Phillips, *Functional Analysis and Semi-Groups*, American Mathematical Society, 1957.
- [16] E.M. Landesman, A.C. Lazer, Nonlinear perturbations of linear elliptic boundary value problems at resonance, *J. Math. Mech.* 19 (1970) 609–623.
- [17] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, 1983.
- [18] A. Schiaffino, K. Schmitt, Periodic solutions of a class of nonlinear evolution equations, *Ann. Mat. Pura Appl.* 137 (1984) 265–272.